



TITLE:

The variety of $\mathfrak{sa}(X)$ (Axiomatic Set Theory and Set- theoretic Topology)

AUTHOR(S):

Yoshinobu, Yasuo

CITATION:

Yoshinobu, Yasuo. The variety of $\mathfrak{sa}(X)$ (Axiomatic Set Theory and Set-theoretic Topology). 数理解析研究所講究録 2008, 1595: 89-96

ISSUE DATE:

2008-04

URL:

<http://hdl.handle.net/2433/81686>

RIGHT:

The variety of $\mathbf{sa}(X)$

吉信康夫 (Yasuo Yoshinobu)*

名古屋大学大学院情報科学研究科

(The Graduate School of Information Science, Nagoya University)

March 13, 2008

In [3], Kada and Tomoyasu defined some cardinal characteristics concerning approximating the Stone-Čech compactification of a metrizable space by a family of its metric-dependent compactifications, and raised several questions on these characteristics. Since then, Kada, Tomoyasu and the author have been studying this subject ([4], [6] and [5]). In this article the author presents a few (rather simple) observations which were obtained in the author's recent study, which was done as a part of this continuing joint research.

1 Basic definitions and backgrounds

For topological spaces X and αX satisfying $X \subseteq \alpha X$, we say αX is a *compactification* of X if αX is compact Hausdorff and X is dense in αX . For compactifications $\alpha X, \gamma X$ of X , we denote $\alpha X \geq_X \gamma X$ if there is a continuous mapping of αX onto γX which is identity on X . We also denote $\alpha X \simeq_X \gamma X$ if $\alpha X \geq_X \gamma X \geq_X \alpha X$ holds, or equivalently there is a homeomorphism between αX and γX which is identity on X . Note that \simeq_X is a (class) equivalent relation on the class $\text{Cpt}(X)$ of compactifications of X , and by identifying \simeq_X -equivalent compactifications we may consider that $\text{Cpt}(X)$ is a set and that \leq_X is a partial ordering of $\text{Cpt}(X)$.

The following are well-known facts about $\text{Cpt}(X)$.

Proposition 1. (1) $\text{Cpt}(X) \neq \emptyset$ iff X is completely regular.

(2) If $\text{Cpt}(X) \neq \emptyset$, $(\text{Cpt}(X), \leq_X)$ forms an upper semi-lattice. In particular, $\text{Cpt}(X)$ has the \leq_X -largest element, the *Stone-Čech compactification* of X , denoted as βX .

*Supported by Grant-in-Aid for Young Scientists (B) 17740053, MEXT.

An important tool to analyze the structure of $(\text{Cpt}(X), \leq_X)$ is a family of Banach algebras of real-valued functions on X . Let $C^*(X)$ denote the set of bounded continuous functions from X to \mathbb{R} . $C^*(X)$ forms a (real) Banach algebra with respect to the uniform norm. A subalgebra (as a Banach algebra) C of $C^*(X)$ is said to be *regular* if for every closed $F \subseteq X$ and $x \in X$ there is $f \in C$ such that $f(x) = 0$ and $f(p) = 1$ for all $p \in F$. Let $\mathcal{R}(X)$ denote the class of regular subalgebras of $C^*(X)$.

For $\alpha X \in \text{Cpt}(X)$ let $C_{\alpha X}$ denote the set of functions in $C^*(X)$ which can be continuously extended to a function on αX . Then $C_{\alpha X} \in \mathcal{R}(X)$ holds. The mapping which maps each $\alpha X \in \text{Cpt}(X)$ to $C_{\alpha X}$ gives an isomorphism between $(\text{Cpt}(X), \leq_X)$ and $(\mathcal{R}(X), \subseteq)$. See [2] for more details.

Now suppose X is a metrizable space, and d is a metric on X which is consistent with the topology of X . The *Smirnov compactification* $u_d X$ of X with respect to d is defined so that

$$C_{u_d X} = \{f \in C^*(X) \mid f \text{ is uniformly continuous with respect to } d\}.$$

Note that if X is totally bounded with respect to d , $u_d X$ is exactly the same as the completion of X with respect to d .

The following theorem shows that the class of Smirnov compactifications of a space is rich enough to “generate” its Stone-Ćech compactification.

Theorem 2. (Woods [8]) For any metrizable space X ,

$$\bigvee_{d \in M(X)} u_d X \simeq_X \beta X$$

holds, where $M(X)$ denotes the set of metrics on X which are consistent with the topology of X , and the join in the left-hand side is taken in the upper semilattice $(\text{Cpt}(X), \leq_X)$.

Inspired with this theorem, Kada and Tomoyasu raised the following general question: For various metrizable spaces, how many metrics do we need to generate their Stone-Ćech compactifications?

Definition 3. (Kada and Tomoyasu [3]; see also [4]) For a metrizable space X , define

$$\text{sa}(X) = \min\{|D| \mid D \subseteq M(X) \wedge \bigvee_{d \in D} u_d X \simeq_X \beta X\}.$$

The following are general facts about $\text{sa}(X)$:

Theorem 4. (Kada and Tomoyasu [3] for (1); Kada, Tomoyasu and Yoshinobu [6] for (2); [5] for (3))

- (1) $\mathfrak{sa}(X) = 1$ holds if and only if the set of non-isolated points of X is compact.
- (2) If $\mathfrak{sa}(X) \neq 1$ then $\mathfrak{sa}(X) \geq \mathfrak{d}$ (the dominating number).
- (3) For an arbitrarily large cardinal θ , there exists a metrizable space X such that $\mathfrak{sa}(X) \geq \theta$.

On the other hand, if X is separable, $\mathfrak{sa}(X) \leq \mathfrak{c}(= 2^{\aleph_0})$ holds since there are at most \mathfrak{c} metrics on X , and thus the problem comes within the range of set theory of reals. The authors have been working on deciding $\mathfrak{sa}(X)$ for various separable X 's.

Theorem 5. (Kada, Tomoyasu and Yoshinobu [6] for (1); [5] for (2), (3))

- (1) $\mathfrak{sa}(X) = \mathfrak{d}$ holds for every non-compact, locally compact separable metrizable space X .
- (2) $\mathfrak{sa}(\mathbb{Q}) = \mathfrak{sa}(\mathbb{R} \setminus \mathbb{Q}) = \mathfrak{d}$.
- (3) $\mathfrak{sa}(\mathbb{B}) = \mathfrak{c}$ for a Bernstein subset \mathbb{B} of \mathbb{R} .

Having these results, the following question was raised in [5].

Question 6. Is it consistent that there exists a separable metrizable space X such that $\mathfrak{d} < \mathfrak{sa}(X) < \mathfrak{c}$?

If X is separable, X can be homeomorphically embedded into the Hilbert cube $\mathbb{H} = {}^\omega[0, 1]$ (with the product topology). So in such cases we regard X as a subspace of \mathbb{H} . Let us denote $X^* = \bar{X} \setminus X$, where \bar{X} denotes the closure of X in \mathbb{H} .

The following theorem, observed independently by Kada and Todorćević, shows that the study of $\mathfrak{sa}(X)$ for a separable metrizable X can be reduced to combinatorics on compact subsets of a separable metrizable space.

Theorem 7. (Kada, Todorćević(see [5])) Suppose $X \subseteq \mathbb{H}$ and $\mathfrak{sa}(X) > 1$. Then the following holds:

$$\mathfrak{sa}(X) = \max\{\mathfrak{d}, \text{cof}(\mathcal{K}(X^*), \subseteq)\},$$

where $\mathcal{K}(X^*)$ denotes the class of compact subsets of X^* .

Note that any separable metrizable space Y is homeomorphic to X^* for some $X \subseteq \mathbb{H}$, since Y can be regarded as a subspace of $\{f \in \mathbb{H} \mid f(0) = 0\}$, which is homeomorphic to \mathbb{H} itself, and thus by letting $X = \mathbb{H} \setminus Y$ we have $Y = X^*$. Therefore Question 6 is equivalent to the following:

Question 8. Is it consistent that there exists a separable metrizable space Y such that $\mathfrak{d} < \text{cof}(\mathcal{K}(Y), \subseteq) < \mathfrak{c}$?

2 ${}^\omega\infty$ -bounding posets and countable compact subsets of metrizable spaces

Here we introduce a property of posets and observe the effect of forcing by posets with this property on the structure of countable compact subsets of metrizable spaces.

Definition 9. Let \mathbb{P} be a poset.

- (1) For an ordinal λ , \mathbb{P} is ${}^\omega\lambda$ -*bounding* if for any $f : \omega \rightarrow \lambda$ in $V^\mathbb{P}$ there exists a function $F : \omega \rightarrow [\lambda]^{<\omega}$ in V such that $\forall n < \omega (f(n) \in F(n))$ holds.
- (2) \mathbb{P} is ${}^\omega\infty$ -*bounding* if \mathbb{P} is ${}^\omega\lambda$ -bounding for all ordinal λ .
- (3) \mathbb{P} is ω -*covering* if whenever X is a countable set of ordinals in $V^\mathbb{P}$ there exists a countable set Y in V such that $X \subseteq Y$.

Note that if \mathbb{P} is ${}^\omega\infty$ -bounding, it is also true that for any $f \in V^\mathbb{P}$ from ω to V , there exists a function $F \in V$ such that $F(n)$ is finite and $f(n) \in F(n)$ for all $n < \omega$.

Lemma 10. Suppose \mathbb{P} is an ${}^\omega\infty$ -bounding poset and X is a metrizable space in V . Then any $C \subseteq X$ in $V^\mathbb{P}$ which is countable and compact in $V^\mathbb{P}$ is covered by some $C_0 \subseteq X$ in V which is countable and compact in V .

Proof. Fix a metric d on X within V . We prove the lemma by induction on the Cantor-Bendixson rank α of C . The case $\alpha = 0$ is trivial, since in this case $C = \emptyset$ holds. Otherwise, argue in $V^\mathbb{P}$ for a while. By the compactness of C , $\alpha = \xi + 1$ for some ξ , and letting F denote the set of points of rank ξ in C , we have F is finite (non-empty) and thus is in V . Pick a positive real $d_0 \in V$ which is larger than the diameter of C (this is possible since C is compact). For each $n < \omega$ let

$$X_n = \{x \in X \mid \frac{d_0}{2^n} \geq d(x, F) \geq \frac{d_0}{2^{n+1}}\}, \text{ and } K_n = C \cap X_n.$$

Note that $\{X_n\}_{n < \omega}$ is defined within V . Note also that each K_n is a closed subset of C and thus is compact, and that $F \cup \bigcup_{n < \omega} K_n = C$ holds. Moreover, the Cantor-Bendixson rank of each K_n is strictly smaller than α , since K_n contains no points in F . Now by the induction hypothesis, for each $n < \omega$ there exists a countable compact $f(n) \in V$ such that $K_n \subseteq f(n) \subseteq X$. Then by the note after Lemma 11 there exists a function $H \in V$ such that $H(n)$ is finite and $f(n) \in H(n)$ holds for all $n < \omega$. Moreover, we may

assume that each $H(n)$ consists only of countable compact subsets of X_n in V . Now let $C_0 = F \cup \bigcup_{n < \omega} H(n)$. It is clear that C_0 is a countable set in V and $C \subseteq C_0$. To see that C_0 is compact, let $\{x_n\}$ be any sequence in C_0 . Then either $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ holds, or, for infinitely many n 's x_n is in some fixed $\bigcup H(m)$, which is a finite union of compact sets and thus is itself compact. In any case, there exists a subsequence of $\{x_n\}$ converging to a point in C_0 . \square

Remark

- (1) The converse of Lemma 10 is also true. For an ordinal λ , let $X = (\omega \times \lambda) \cup \{(\omega, 0)\}$ and define a metric d on X as follows: for every two distinct $(m, \alpha), (n, \beta) \in X$ let $d((m, \alpha), (n, \beta)) = 2^{-\min\{m, n\}}$. Note that for each $f : \omega \rightarrow \lambda$, $C_f = \{(n, f(n)) \mid n < \omega\} \cup \{(\omega, 0)\}$ is a (countable) compact subset of X , and that for any compact subset C of X , $F(n) = \{\alpha < \lambda \mid (n, \alpha) \in C\}$ is finite for all $n < \omega$. Using these facts one can show that the conclusion of Lemma 10 for this X implies that \mathbb{P} is ${}^\omega\lambda$ -bounding.
- (2) One can also show that for the conclusion of Lemma 10 only for separable metrizable spaces, the assumption that \mathbb{P} is ${}^\omega\mathfrak{c}$ -bounding is sufficient (and necessary).

The following is a useful criterion for a poset to be ${}^\omega\infty$ -bounding.

Lemma 11. For a poset \mathbb{P} , the following are equivalent:

- (a) \mathbb{P} is ${}^\omega\infty$ -bounding,
- (b) \mathbb{P} is ${}^\omega\omega$ -bounding and ω -covering.

Proof. ((a) \Rightarrow (b)) Assume (a). It is enough to show that \mathbb{P} is ω -covering. Let X be a countable set of ordinals in $V^{\mathbb{P}}$. Fix a surjection $f : \omega \rightarrow X$. By (a) there exists a function $F \in V$ on ω such that $F(n)$ is finite and $f(n) \in F(n)$ holds for all $n < \omega$. Thus $\bigcup \text{range}(F)$ is countable in V and contains X .

((b) \Rightarrow (a)) Assume (b). Let $f \in V^{\mathbb{P}}$ be any ordinal-valued function on ω . Since $\text{range}(f)$ is a countable set of ordinals in $V^{\mathbb{P}}$ and \mathbb{P} is ω -covering, there exists a countable set Y in V such that $\text{range}(f) \subseteq Y$. Now let $\{y_m\}_{m < \omega}$ be an enumeration of Y in V , and define $g : \omega \rightarrow \omega$ so that $f(n) = y_{g(n)}$ holds for each $n < \omega$. Since $g \in V^{\mathbb{P}}$ and \mathbb{P} is ${}^\omega\omega$ -covering, there exists a function $G : \omega \rightarrow \omega$ in V such that $g(n) < G(n)$ for all $n < \omega$. Define F so that $F(n) = \{y_m \mid m < G(n)\}$ for each $n < \omega$. Then $F \in V$ and for all $n < \omega$ $F(n)$ is finite and $f(n) \in F(n)$ holds. \square

3 Consistency of $\mathfrak{d} < \mathfrak{sa}(X) < \mathfrak{c}$

For an infinite cardinal κ , let $\mathbb{B}(\kappa)$ denote the measure algebra on κ . It is well-known that $\mathbb{B}(\kappa)$ is ${}^\omega\omega$ -bounding (see [1]), and is also ω -covering since it satisfies the countable chain condition. Thus by Lemma 11 $\mathbb{B}(\kappa)$ is ${}^\omega\infty$ -bounding. The following theorem gives an affirmative answer to Question 8 and thus to Question 6 in a strong sense.

Theorem 12. Assume GCH and κ is a cardinal in V . Then in $V^{\mathbb{B}(\kappa)}$, for every cardinal θ satisfying $\mathfrak{d}(= \aleph_1) \leq \theta \leq \mathfrak{c}$ there exists a separable metrizable space X such that $\mathfrak{sa}(X) = \theta$.

Proof. We may assume $\mathfrak{d} < \theta < \mathfrak{c}$, since the cases $\theta = \mathfrak{d}$ and $\theta = \mathfrak{c}$ are already done (Theorem 5). Under this assumption we have $\theta \leq \kappa$. By Theorem 7 it is enough to show that there exists a set $Y \subseteq \mathbb{H}$ such that $\text{cof}(\mathcal{K}(Y), \subseteq) = \theta$.

Case 1 $\text{cf}\theta \neq \omega$.

We will use the fact that $\mathbb{B}(\kappa)$ can be factorized as $\mathbb{B}(\theta) * \dot{\mathbb{B}}(\kappa \setminus \theta)$. Argue in $V^{\mathbb{B}(\kappa)}$. Let $Y = \mathbb{H} \cap V^{\mathbb{B}(\theta)}$. Note that $|Y| = \mathfrak{c}^{V^{\mathbb{B}(\theta)}} = \theta < \mathfrak{c}$, and thus any compact subset of Y is at most countable. This shows that $\text{cof}(\mathcal{K}(Y), \subseteq) \geq \theta$, since any \subseteq -cofinal subfamily of $\mathcal{K}(Y)$ must cover Y . On the other hand, let \mathcal{F} be the family of countable compact subsets of \mathbb{H} computed in $V^{\mathbb{B}(\theta)}$. Then $|\mathcal{F}| = \theta$, and since $V^{\mathbb{B}(\kappa)}$ is an ${}^\omega\infty$ -bounding extension of $V^{\mathbb{B}(\theta)}$, by Lemma 10, \mathcal{F} remains to be a \subseteq -cofinal subfamily of $\mathcal{K}(Y)$ in $V^{\mathbb{B}(\kappa)}$ ¹. This shows that $\text{cof}(\mathcal{K}(Y), \subseteq) \leq \theta$.

Case 2 $\text{cf}\theta = \omega$.

Argue in $V^{\mathbb{B}(\kappa)}$ again. Let $\{\theta_n\}_{n < \omega}$ be regular uncountable cardinals such that $\sup_{n < \omega} \theta_n = \theta$. Case 1 shows that for each $n < \omega$ there exists a set $Y_n \subseteq \mathbb{H}$ such that $|Y_n| = \theta_n$ and $\text{cof}(\mathcal{K}(Y_n), \subseteq) = \theta_n$. Let \mathcal{K}_n be a \subseteq -cofinal subfamily of $\mathcal{K}(Y_n)$ such that $|\mathcal{K}_n| = \theta_n$. Now let

$$Y = \{f \in \mathbb{H} \mid \exists n < \omega (f(0) = \frac{1}{n} \wedge \bar{f} \in Y_n)\},$$

where $\bar{f} \in \mathbb{H}$ is defined by $\bar{f}(n) = f(n+1)$ ($\forall n < \omega$) for $f \in \mathbb{H}$. Then $|Y| = \theta$ and by the same argument as in Case 1 we have $\text{cof}(\mathcal{K}(Y), \subseteq) \geq \theta$. Now note that any compact subset of Y is of the form $\bigcup_{n < m} K_n$ for some $m < \omega$, where each K_n is a compact subset of $Y'_n = \{f \in Y \mid f(0) = \frac{1}{n}\}$. Therefore

$$\mathcal{K} = \left\{ \bigcup_{n < m} K_n \mid m < \omega \wedge \forall n < m (\{\bar{f} \mid f \in K_n\} \in \mathcal{K}_n) \right\}$$

¹Here we used the fact that the compactness of a countable metric space is upward-absolute. This follows from the fact that the compactness of a countable metric space is a Π^1_1 -statement about its metric.

forms a \subseteq -cofinal subfamily of $\mathcal{K}(Y)$. It is easy to check that $|\mathcal{K}| = \theta$ holds. This shows that $\text{cof}(\mathcal{K}(Y), \subseteq) \leq \theta$. \square

After having the above observation, the author noticed that older studies had already suggested that if $\theta \in [\mathfrak{d}, \mathfrak{c}]$ is small enough, it is always the case that there exists a separable metrizable space X such that $\mathfrak{sa}(X) = \theta$.

Theorem 13. (van Douwen [7, Theorem 8.10(a), (b)]) For a countable metric space X , $\text{cof}(\mathcal{K}(X), \subseteq) \leq \mathfrak{d}$ holds.

Corollary 14. For any cardinal θ satisfying $\mathfrak{d} \leq \theta \leq \min\{\aleph_\omega, \mathfrak{c}\}$ there exists a separable metrizable X such that $\mathfrak{sa}(X) = \theta$.

Proof. Again we may assume $\mathfrak{d} < \theta < \mathfrak{c}$, and it is enough to show that there exists a set $Y \subseteq \mathbb{H}$ such that $\text{cof}(\mathcal{K}(Y), \subseteq) = \theta$.

Case 1 $\theta = \aleph_n$ for some $n < \omega$ (thus in fact $n > 1$).

First note that $[\theta]^{\leq \aleph_0}$ has a \subseteq -cofinal subfamily of size θ : ω_1 is a \subseteq -cofinal subfamily of $[\omega_1]^{\leq \aleph_0}$ of size ω_1 . If K_m is a \subseteq -cofinal subfamily of $[\omega_m]^{\leq \aleph_0}$ of size \aleph_m , then it is easy to see that $\bigcup_{\gamma < \omega_{m+1}} (f_\gamma^m)'' K_m$ is a \subseteq -cofinal subfamily of $[\omega_{m+1}]^{\leq \aleph_0}$ of size \aleph_{m+1} (where f_γ^m denotes a surjection from ω_m to γ).

Pick any $Y \subseteq \mathbb{H}$ of size θ . Since every compact subset of Y is countable, by the same argument as in the proof of Theorem 12, $\text{cof}(\mathcal{K}(Y), \subseteq) \geq \theta$ holds. On the other hand, letting \mathcal{C} be a \subseteq -cofinal subfamily of $[Y]^{\leq \aleph_0}$ of size θ , by Theorem 13 we have

$$\text{cof}(\mathcal{K}(Y), \subseteq) \leq \sum_{X \in \mathcal{C}} \text{cof}(\mathcal{K}(X), \subseteq) \leq |\mathcal{C}| \cdot \mathfrak{d} = \theta.$$

Case 2 $\theta = \aleph_\omega$.

This case can be dealt with in exactly the same way as in Case 2 in the proof of Theorem 12, using Case 1. \square

Having these results, our next question becomes of the following kind, somewhat with an opposite tone to Question 6.

Question 15. Is it consistent that $\mathfrak{d} < \aleph_{\omega+1} < \mathfrak{c}$ no separable metrizable space X satisfies $\mathfrak{sa}(X) = \aleph_{\omega+1}$? More generally, is it consistent that there exists a cardinal θ such that $\mathfrak{d} < \theta < \mathfrak{c}$ with no separable metrizable space X such that $\mathfrak{sa}(X) = \theta$?

References

- [1] T. Bartoszyński and H. Judah. *Set theory on the structure of the real line*. A K Peters, 1995.

- [2] L. Gillman and M. Jerison. *Rings of continuous functions*. Van Nostrand, 1960.
- [3] M. Kada and K. Tomoyasu. How many miles to $\beta\omega$? (General and Geometric Topology and Related Topics). *RIMS Kokyuroku*, 1370:86–101, 2004. (Japanese).
- [4] M. Kada, K. Tomoyasu, and Y. Yoshinobu. How many miles to $\beta\omega$? — Approximating $\beta\omega$ by metric-dependent compactifications. *Topology Appl.*, 145:277–292, 2004.
- [5] M. Kada, K. Tomoyasu, and Y. Yoshinobu. How many miles to $\beta\omega$? II (Set Theoretic and Geometric Topology and Its Applications). *RIMS Kokyuroku*, 1419:105–125, 2005. (Japanese).
- [6] M. Kada, K. Tomoyasu, and Y. Yoshinobu. How many miles to βX ? — \mathfrak{d} miles, or just one foot. *Topology Appl.*, 153:3313–3319, 2006.
- [7] E. K. van Douwen. The integers and topology. In K. Kunen and J. E. Vaughan, editors, *Handbook of Set Theoretic Topology*, pages 111–167. North-Holland, 1984.
- [8] R.G. Woods. The minimum uniform compactification of a metric space. *Fund. Math.*, 147:39–59, 1995.